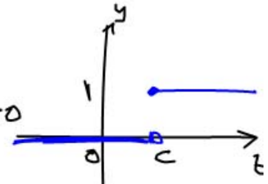
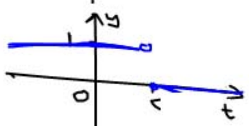
6.3 Adım Fonksiyonları

$$u_c(t)=\begin{cases}0, & t<c\\ 1, & t\ge c\end{cases} \qquad c\ge 0$$



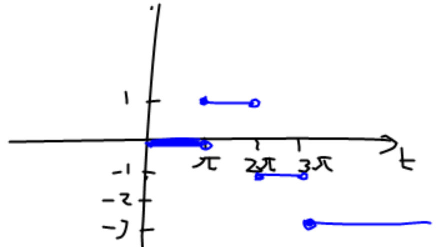
fonksiyonuna birim adım fonksiyonu veya Heaviside fonksiyonu denir.

$$y=1-u_c(t)=\begin{cases}1, & t<c\\ 0, & t\ge c\end{cases}$$



Örnek: $h(t)=u_\pi(t)-2u_{2\pi}(t)-2u_{3\pi}(t)$, $t\ge 0$ fonksiyonunun grafiğini çiziniz.

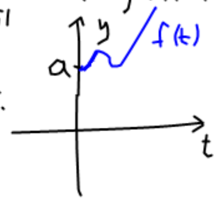
$$h(t)=\begin{cases}0-2.0-2.0=0, & 0\le t<\pi\\ 1-2.0-2.0=1, & \pi\le t<2\pi\\ 1-2.1-2.0=-1, & 2\pi\le t<3\pi\\ 1-2.1-2.1=-3, & t\ge 3\pi\end{cases}$$

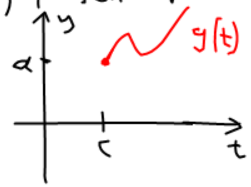


$t\ge 0$ olmak üzere, verilen bir f fonksiyonu için g fonksiyonunu

$$y=g(t)=\begin{cases}0, & t<c\\ f(t-c), & t\ge c\end{cases}$$

ile tanımlıyalım. g, f fonksiyonunu pozitif t yönünde c uzaklığı kadar öteler.





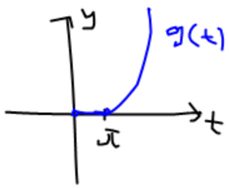
$g(t)$'yi birim adım fonksiyonu yardımıyla yazalım.

$$g(t)=u_c(t)\,f(t-c)$$

Örnekler ($t\ge 0$ alıyoruz) Aşağıdaki fonkları açık şekilde yazınız ve grafiklerini çiziniz.

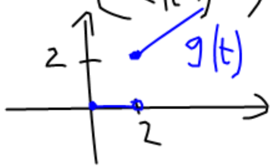
1) $f(t)=t^2$, $g(t)=u_\pi(t)\,f(t-\pi)$

$$g(t)=\begin{cases}0, & 0\le t<\pi\\ (t-\pi)^2, & t\ge\pi\end{cases}$$



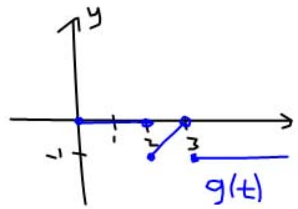
2) $f(t)=2t$, $g(t)=f(t-1)\,u_2(t)$

$$g(t)=\begin{cases}0, & 0\le t<2\\ 2(t-1), & t\ge 2\end{cases}$$



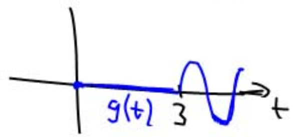
3) $g(t)=(t-3)\,u_2(t)-(t-2)\,u_3(t)$

$$g(t)=\begin{cases}0, & 0\le t<2\\ t-3, & 2\le t<3\\ -1, & t\ge 3\end{cases}$$



4) $f(t)=\sin t$ $\quad g(t)=f(t-3)\,u_3(t)$

$$g(t)=\begin{cases}0, & 0\le t<3\\ \sin(t-3), & t\ge 3\end{cases}$$



$$\mathcal{L}\{u_c(t)\}=\int_0^\infty e^{-st}u_c(t)\,dt=\int_c^\infty e^{-st}\,dt=\frac{e^{-cs}}{s},\quad s>0$$

Teorem: Eğer $s \geq a > 0$ için $F(s)=\mathcal{L}\{f(t)\}$ var ve c pozitif bir sayı ise

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s>a$$

dir. Tersine eğer $f(t)=\mathcal{L}^{-1}\{F(s)\}$ ise

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

dir.

Kanıt:

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st} f(t-c)\,dt$$

$$\begin{bmatrix} t-c=u \\ dt=du \end{bmatrix} = \int_0^\infty e^{-s(u+c)} f(u)\,du$$

$$= e^{-cs}\int_0^\infty e^{-su}f(u)\,du$$

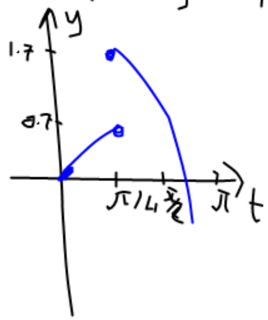
$$= e^{-cs}F(s), \quad s>a$$

Örnek: 1) $f(t)=1$, $\mathcal{L}\{1\}=\frac{1}{s}$, $s>0$

$$\mathcal{L}\{u_c(t)\} = e^{-cs}\mathcal{L}\{1\} = \frac{e^{-cs}}{s}, \quad s>0$$

2) $f(t) = \begin{cases} \sin t, & 0 \leq t < \frac{\pi}{4} \\ \sin t + \cos(t-\pi/4), & t \geq \frac{\pi}{4} \end{cases}$ ise

$$\mathcal{L}\{f(t)\} = ?$$



$$f(t) = \sin t + u_{\pi/4}(t)\cos(t-\tfrac{\pi}{4})$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + e^{-\frac{\pi}{4}s}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2+1} + \frac{s\,e^{-\pi/4\, s}}{s^2+1}$$

$$= \frac{1+s\,e^{-\pi/4\, s}}{s^2+1}$$

3) $f(t) = \begin{cases} 0, & 0 \leq t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$, $\mathcal{L}\{f(t)\}=?$

$$f(t) = u_2(t)(t-2)^2$$

$$\mathcal{L}\{f(t)\} = e^{-2s}\mathcal{L}\{t^2\} = \frac{2e^{-2s}}{s^3}, \quad s>0$$

4) $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$, $\mathcal{L}\{f(t)\}=?$

$$f(t) = u_2(t)(t-2-1) - u_3(t)(t-3+1)$$

$$\mathcal{L}\{f(t)\} = e^{-2s}\mathcal{L}\{t-1\} - e^{-3s}\mathcal{L}\{t+1\}$$

$$= e^{-2s}\left(\frac{1}{s^2}-\frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s^2}+\frac{1}{s}\right), \quad s>0$$

5) $F(s) = \dfrac{1-e^{-2s}}{s^2}$ ters Laplace dönüşümünü bul.

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}\left\{\frac{1}{s^2}\right\} - \mathcal{L}\left\{e^{-2s}\frac{1}{s^2}\right\} = t - u_2(t)(t-2)$$

$$= \begin{cases} t, & 0 \leq t < 2 \\ 2, & t \geq 2 \end{cases}$$

Teorem: Eğer $s \geq a > 0$ için $F(s)=\mathcal{L}\{f(t)\}$ var ve c sabit bir sayı ise

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s>a+c$$

dir. Tersine $f(t)=\mathcal{L}^{-1}\{F(s)\}$ var ise

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

dir.

Kanıt: $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt$

$= F(s-c)$, $(s > a+c$ teoremden)

Örnek: $G(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+2}$ ise $\mathcal{L}^{-1}\{G(s)\}=?$

$G(s) = \frac{2(s-1)e^{-2(s-1)}\cdot e^{-2}}{(s-1)^2+1} = 2e^{-2}F(s-1)$ $\quad F(s)=\frac{se^{-2s}}{s^2+1}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = u_2(t)\cos(t-2)$

$\mathcal{L}^{-1}\{G(s)\} = 2e^{-2}\mathcal{L}^{-1}\{F(s-1)\} = 2e^{-2}\cdot e^{t}u_2(t)\cos(t-2)$

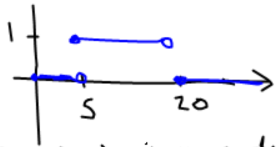
$= 2e^{t-2}u_2(t)\cos(t-2)$

6.4 Süreksiz kuvvet Fonksiyonlarına Sahip Dif. Denklemler

Örnek: $2y''+y'+2y=g(t)$ $\quad y(0)=y'(0)=0$

$g(t)=u_5(t)-u_{20}(t)$ başlangıç değer problemini çöz.

$$g(t)=\begin{cases}0, & 0\le t<5\\ 1, & 5\le t<20\\ 0, & t\ge 20\end{cases}$$



($0\le t<5$ aralığında bu denklemin çözümü $y=0$ dır. $t\ge 20$ için bu denklemin çözümü $y=c_1e^{-t/4}\cos\frac{\sqrt{15}}{4}t + c_2e^{-t/4}\sin\frac{\sqrt{15}}{4}t$ dir.)

$2\mathcal{L}\{y''\}+\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{u_5(t)\}-\mathcal{L}\{u_{20}(t)\}$

$2\{s^2Y(s)-sy(0)-y'(0)\}+sY(s)-y(0)+2Y(s)=\frac{e^{-5s}}{s}-\frac{e^{-20s}}{s}$

$(2s^2+s+2)Y(s)=\frac{e^{-5s}-e^{-20s}}{s}$

$Y(s)=\frac{1}{s(2s^2+s+2)}\left(e^{-5s}-e^{-20s}\right)$ $\quad A=\frac{1}{2}$

$H(s)=\frac{1}{s(2s^2+s+2)}=\frac{A}{s}+\frac{Bs+C}{2s^2+s+2}$ $\quad B=-1 \quad C=-\frac{1}{2}$

$=\frac{1/2}{s}+\frac{-s-\frac{1}{2}}{2s^2+s+2}$

$=\frac{1/2}{s}-\frac{1}{2}\,\frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}$

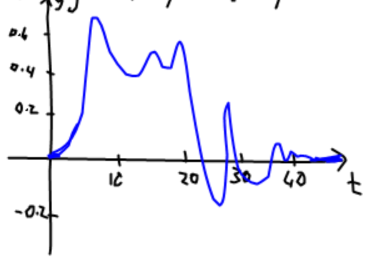
$h(t)=\mathcal{L}^{-1}\{H(s)\}=\frac{1}{2}-\frac{1}{2}\Big\{e^{-t/4}\cos\frac{\sqrt{15}}{4}t+$

$+\frac{\sqrt{15}}{15}e^{-t/4}\sin\frac{\sqrt{15}}{4}t\Big\}$

$Y(s)=H(s)\left(e^{-5s}-e^{-20s}\right)$

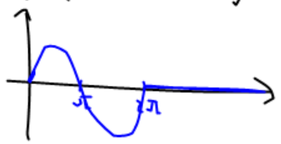
$y(t)=\mathcal{L}^{-1}\{Y(s)\}=u_5(t)\,h(t-5)-u_{20}(t)\,h(t-20)$

y'nin grafiği aşağıdaki gibidir.



2) $y''+4y=\begin{cases}\sin t, & 0\le t<2\pi\\ \sin t-\sin(t-2\pi), & t\ge 2\pi\end{cases}$

$y(0)=y'(0)=0$ başlangıç değer prob. çöz. ve çözümün grafiğini çiziniz.



$$y''+4y=\sin t-u_{2\pi}(t)\sin(t-2\pi)$$

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\mathcal{L}\{\sin t\}-\mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\}$$

$$(s^2+4)Y(s)=\frac{1}{s^2+1}-\frac{e^{-2\pi s}}{s^2+1}$$

$$Y(s)=\frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

$$H(s)=\frac{1}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

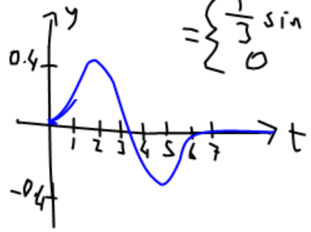
$$=\frac{1/3}{s^2+1}-\frac{1/3}{s^2+4}$$

$$h(t)=\mathcal{L}^{-1}\{H(s)\}=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t$$

$$Y(s)=H(s)\left[1-e^{-2\pi s}\right]$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=h(t)-u_{2\pi}(t)\,h(t-2\pi)$$

$$=\begin{cases}\frac{1}{3}\sin t-\frac{1}{6}\sin 2t, & 0\le t<2\pi\\ 0, & t\ge 2\pi\end{cases}$$



6.5 Darbe (impuls) Fonksiyonları

Bazı uygulamalarda çok kısa aralıkta, büyük değerlere sahip, örneğin voltaj gibi, fiziksel olaylarla ilgilenmek zorunda kalınılır. Bu problemler sıklıkla

$$ay''+by'+cy=g(t)$$

formundaki dif. denklemler ile temsil edilirler. Burada $g(t)$ bir $t_0-\tau<t<t_0+\tau$ çok kısa aralığında büyük değere sahip diğer yerlerde sıfır olan bir fonksiyondur.